3.2.11 Theorem *Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_{n+1}/x_n)$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$.*

Proof. By 3.2.4 it follows that $L \geq 0$. Let r be a number such that $L < r < 1$, and let $\varepsilon := r - L > 0$. There exists a number $K \in \mathbb{N}$ such that if $n \geq K$ then

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon.$$

It follows from this (why?) that if $n \geq K$, then

$$\frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r.$$

Therefore, if $n \geq K$, we obtain

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_K r^{n-K+1}.$$

If we set $C := x_K/r^K$, we see that $0 < x_{n+1} < Cr^{n+1}$ for all $n \geq K$. Since $0 < r < 1$, it follows from 3.1.11(b) that $\lim(r^n) = 0$ and therefore from Theorem 3.1.10 that $\lim(x_n) = 0$. Q.E.D.

As an illustration of the utility of the preceding theorem, consider the sequence (x_n) given by $x_n := n/2^n$. We have

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

so that $\lim(x_{n+1}/x_n) = \frac{1}{2}$. Since $\frac{1}{2} < 1$, it follows from Theorem 3.2.11 that $\lim(n/2^n) = 0$.

Exercises for Section 3.2

1. For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = (x_n)$.
   (a) $x_n := \dfrac{n}{n+1}$, (b) $x_n := \dfrac{(-1)^n n}{n+1}$,
   (c) $x_n := \dfrac{n^2}{n+1}$, (d) $x_n := \dfrac{2n^2+3}{n^2+1}$.
2. Give an example of two divergent sequences X and Y such that:
   (a) their sum $X + Y$ converges, (b) their product XY converges.
3. Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.
4. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.
5. Show that the following sequences are not convergent.
   (a) (2^n), (b) $((-1)^n n^2)$.
6. Find the limits of the following sequences:
   (a) $\lim\left((2 + 1/n)^2\right)$, (b) $\lim\left(\dfrac{(-1)^n}{n+2}\right)$,
   (c) $\lim\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, (d) $\lim\left(\dfrac{n+1}{n\sqrt{n}}\right)$.

7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 *cannot* be used.
8. Explain why the result in equation (3) before Theorem 3.2.4 *cannot* be used to evaluate the limit of the sequence $((1 + 1/n)^n)$.
9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}\, y_n)$ converges. Find the limit.
10. Determine the limits of the following sequences.
    (a) $(\sqrt{4n^2 + n} - 2n)$, (b) $(\sqrt{n^2 + 5n} - n)$.
11. Determine the following limits.
    (a) $\lim\left((3\sqrt{n})^{1/2n}\right)$, (b) $\lim\left((n+1)^{1/\ln(n+1)}\right)$.
12. If $0 < a < b$, determine $\lim\left(\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}\right)$.
13. If $a > 0, b > 0$, show that $\lim\left(\sqrt{(n+a)(n+b)} - n\right) = (a+b)/2$.
14. Use the Squeeze Theorem 3.2.7 to determine the limits of the following,
    (a) (n^{1/n^2}), (b) $\left((n!)^{1/n^2}\right)$.
15. Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.
16. Apply Theorem 3.2.11 to the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) (a^n), (b) $(b^n/2^n)$,
    (c) (n/b^n), (d) $(2^{3n}/3^{2n})$.
17. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_{n+1}/x_n) = 1$.
    (b) Give an example of a divergent sequence with this property. (Thus, this property cannot be used as a test for convergence.)
18. Let $X = (x_n)$ be a sequence of positive real numbers such that $\lim(x_{n+1}/x_n) = L > 1$. Show that X is not a bounded sequence and hence is not convergent.
19. Discuss the convergence of the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) $(n^2 a^n)$, (b) (b^n/n^2),
    (c) $(b^n/n!)$, (d) $(n!/n^n)$.
20. Let (x_n) be a sequence of positive real numbers such that $\lim(x_n^{1/n}) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $\lim(x_n) = 0$.
21. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$.
    (b) Give an example of a divergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$. (Thus, this property cannot be used as a test for convergence.)
22. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?
23. Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent. (See Exercise 2.2.18.)
24. Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \operatorname{mid}\{x_n, y_n, z_n\}$ is also convergent. (See Exercise 2.2.19.)

Section 3.3 Monotone Sequences

Until now, we have obtained several methods of showing that a sequence $X = (x_n)$ of real numbers is convergent:

In 1741, Euler accepted an offer from Frederick the Great to join the Berlin Academy, where he stayed for 25 years. During this period he wrote landmark books on a relatively new subject called calculus and a steady stream of papers on mathematics and science. In response to a request for instruction in science from the Princess of Anhalt-Dessau, he wrote her nearly 200 letters on science that later became famous in a book titled *Letters to a German Princess*. When Euler lost vision in one eye, Frederick thereafter referred to him as his mathematical "cyclops."

In 1766, he happily returned to Russia at the invitation of Catherine the Great. His eyesight continued to deteriorate and in 1771 he became totally blind following an eye operation. Incredibly, his blindness made little impact on his mathematics output, for he wrote several books and over 400 papers while blind. He remained active until the day of his death.

Euler's productivity was remarkable. He wrote textbooks on physics, algebra, calculus, real and complex analysis, and differential geometry. He also wrote hundreds of papers, many winning prizes. A current edition of his collected works consists of 74 volumes.

Exercises for Section 3.3

1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.
2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. show that (x_n) is bounded and monotone. Find the limit.
3. Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.
4. Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.
5. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. [*Hint:* One upper bound is $1 + 2\sqrt{p}$.]
6. Let $a > 0$ and let $z_1 > 0$. Define $z_{n+1} := \sqrt{a + z_n}$ for $n \in \mathbb{N}$. Show that (z_n) converges and find the limit.
7. Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine whether (x_n) converges or diverges.
8. Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.
9. Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(x_n)$.
10. Establish the convergence or the divergence of the sequence (y_n), where
$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \quad \text{for} \quad n \in \mathbb{N}.$$
11. Let $x_n := 1/1^2 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges. [*Hint:* Note that if $k \geq 2$, then $1/k^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$.]
12. Establish the convergence and find the limits of the following sequences.
    (a) $\left((1+1/n)^{n+1}\right)$, (b) $\left((1+1/n)^{2n}\right)$,
    (c) $\left(\left(1 + \frac{1}{n+1}\right)^n\right)$, (d) $\left((1-1/n)^n\right)$.
13. Use the method in Example 3.3.5 to calculate $\sqrt{2}$, correct to within 4 decimals.
14. Use the method in Example 3.3.5 to calculate $\sqrt{5}$, correct to within 5 decimals.
15. Calculate the number e_n in Example 3.3.6 for $n = 2, 4, 8, 16$.
16. Use a calculator to compute e_n for $n = 50$, $n = 100$, and $n = 1000$.

(d) implies (a). Let $w = \sup S$. If $\varepsilon > 0$ is given, then there are at most finitely many n with $w + \varepsilon < x_n$. Therefore $w + \varepsilon$ belongs to V and $\limsup (x_n) \leq w + \varepsilon$. On the other hand, there exists a subsequence of (x_n) converging to some number larger than $w - \varepsilon$, so that $w - \varepsilon$ is not in V, and hence $w - \varepsilon \leq \limsup (x_n)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $w = \limsup (x_n)$. Q.E.D.

As an instructive exercise, the reader should formulate the corresponding theorem for the limit inferior of a bounded sequence of real numbers.

3.4.12 Theorem A bounded sequence (x_n) is convergent if and only if $\limsup (x_n) = \liminf (x_n)$.

We leave the proof as an exercise. Other basic properties can also be found in the exercises.

Exercises for Section 3.4

1. Give an example of an unbounded sequence that has a convergent subsequence.
2. Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{1/n}) = 1$.
3. Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.
4. Show that the following sequences are divergent.
   (a) $(1 - (-1)^n + 1/n)$, (b) $(\sin n\pi/4)$.
5. Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_1 := x_1, z_2 := y_1, \ldots, z_{2n-1} := x_n, z_{2n} := y_n, \ldots$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.
6. Let $x_n := n^{1/n}$ for $n \in \mathbb{N}$.
   (a) Show that $x_{n+1} < x_n$ if and only if $(1 + 1/n)^n < n$, and infer that the inequality is valid for $n \geq 3$. (See Example 3.3.6.) Conclude that (x_n) is ultimately decreasing and that $x := \lim(x_n)$ exists.
   (b) Use the fact that the subsequence (x_{2n}) also converges to x to conclude that $x = 1$.
7. Establish the convergence and find the limits of the following sequences:
   (a) $\left((1 + 1/n^2)^{n^2}\right)$, (b) $((1 + 1/2n)^n)$,
   (c) $\left((1 + 1/n^2)^{2n^2}\right)$, (d) $((1 + 2/n)^n)$.
8. Determine the limits of the following.
   (a) $((3n)^{1/2n})$, (b) $((1 + 1/2n)^{3n})$.
9. Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.
10. Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n := \sup\{x_k : k \geq n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.
11. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that (x_n) converges.
12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.
13. If $x_n := (-1)^n/n$, find the subsequence of (x_n) that is constructed in the second proof of the Bolzano-Weierstrass Theorem 3.4.8, when we take $I_1 := [-1, 1]$.

14. Let (x_n) be a bounded sequence and let $s := \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s.
15. Let (I_n) be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to give a proof of the Nested Intervals Property 2.5.2.
16. Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequence is dropped.
17. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \ldots).$$

Determine the values of $\lim \sup(x_n)$ and $\lim \inf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.
18. Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\lim \sup(x_n) = \lim \inf(x_n)$.
19. Show that if (x_n) and (y_n) are bounded sequences, then

$$\lim \sup(x_n + y_n) \le \lim \sup(x_n) + \lim \sup(y_n).$$

Give an example in which the two sides are not equal.

Section 3.5 The Cauchy Criterion

The Monotone Convergence Theorem is extraordinarily useful and important, but it has the significant drawback that it applies only to sequences that are monotone. It is important for us to have a condition implying the convergence of a sequence that does not require us to know the value of the limit in advance, and is not restricted to monotone sequences. The Cauchy Criterion, which will be established in this section, is such a condition.

3.5.1 Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \ge H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

The significance of the concept of Cauchy sequence lies in the main theorem of this section, which asserts that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. This will give us a method of proving a sequence converges without knowing the limit of the sequence.

However, we will first highlight the definition of Cauchy sequence in the following examples.

3.5.2 Examples **(a)** The sequence $(1/n)$ is a Cauchy sequence.

If $\varepsilon > 0$ is given, we choose a natural number $H = H(\varepsilon)$ such that $H > 2/\varepsilon$. Then if $m, n \ge H$, we have $1/n \le 1/H < \varepsilon/2$ and similarly $1/m < \varepsilon/2$. Therefore, it follows that if $m, n \ge H$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \le \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(1/n)$ is a Cauchy sequence.

(b) The sequence $(1 + (-1)^n)$ is *not* a Cauchy sequence.

The negation of the definition of Cauchy sequence is: There exists $\varepsilon_0 > 0$ such that for every H there exist at least one $n > H$ and at least one $m > H$ such that $|x_n - x_m| \ge \varepsilon_0$. For

To estimate the accuracy, we note that $|x_2 - x_1| < 0.2$. Thus, after n steps it follows from Corollary 3.5.10(i) that we are sure that $|x^* - x_n| \le 3^{n-1}(7^{n-2} \cdot 20)$. Thus, when $n = 6$, we are sure that

$$|x^* - x_6| \le 3^5/(7^4 \cdot 20) = 243/48\,020 < 0.0051.$$

Actually the approximation is substantially better than this. In fact, since $|x_6 - x_5| < 0.000\,0005$, it follows from 3.5.10(ii) that $|x^* - x_6| \le \frac{3}{4}|x_6 - x_5| < 0.000\,0004$. Hence the first five decimal places of x_6 are correct. □

Exercises for Section 3.5

1. Give an example of a bounded sequence that is not a Cauchy sequence.
2. Show directly from the definition that the following are Cauchy sequences.
   (a) $\left(\dfrac{n+1}{n}\right)$, (b) $\left(1 + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}\right)$.
3. Show directly from the definition that the following are not Cauchy sequences.
   (a) $((-1)^n)$, (b) $\left(n + \dfrac{(-1)^n}{n}\right)$, (c) $(\ln n)$
4. Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.
5. If $x_n := \sqrt{n}$, show that (x_n) satisfies $\lim|x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.
6. Let p be a given natural number. Give an example of a sequence (x_n) that is not a Cauchy sequence, but that satisfies $\lim|x_{n+p} - x_n| = 0$.
7. Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.
8. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.
9. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.
10. If $x_1 < x_2$ are arbitrary real numbers and $x_n := \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n > 2$, show that (x_n) is convergent. What is its limit?
11. If $y_1 < y_2$ are arbitrary real numbers and $y_n := \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2}$ for $n > 2$, show that (y_n) is convergent. What is its limit?
12. If $x_1 > 0$ and $x_{n+1} := (2 + x_n)^{-1}$ for $n \ge 1$, show that (x_n) is a contractive sequence. Find the limit.
13. If $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \ge 1$, show that (x_n) is a contractive sequence. What is its limit?
14. The polynomial equation $x^3 - 5x + 1 = 0$ has a root r with $0 < r < 1$. Use an appropriate contractive sequence to calculate r within 10^{-4}.

Section 3.6 Properly Divergent Sequences

For certain purposes it is convenient to define what is meant for a sequence (x_n) of real numbers to "tend to $\pm\infty$."